Quantum Double models

A spin-like Hilbert space $\mathcal{V}$ is defined at each lattice link : $\{|g\rangle : g \in G\}$

$$\dim(\mathcal{V}) = |G|$$

Define linear operators $L^g_\pm$, $g \in G$ associated with vertices, and $T^h_\pm$, $h \in G$, associated with plaquettes :

$$L^g_+ |z\rangle = |gz\rangle \,, \quad L^g_- |z\rangle = |zg^{-1}\rangle \,,$$

$$T^h_+ |z\rangle = \delta_{h,z} |z\rangle, \quad T^h_- |z\rangle = \delta_{h^{-1},z} |z\rangle$$

In the case of Toric code model:

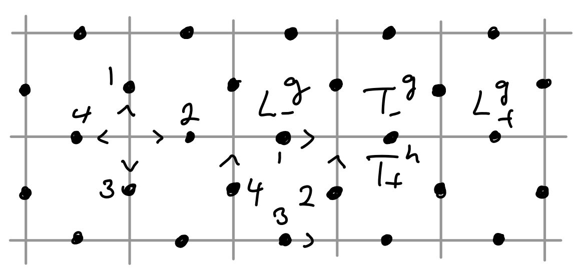
$$L^g_\pm \longleftrightarrow \sigma^x \,, \quad T^h_\pm \longleftrightarrow (\mathbb{1} \pm \sigma^z)/2$$

We have commutation relations:

$$L^g_+ T^h_+ = T^{gh}_+ L^g_+ \,, \quad L^g_- T^h_+ = T^{hg^{-1}}_+ L^g_- \,,$$

$$L^g_+ T^h_- = T^{hg^{-1}}_- L^g_+ \,, \quad L^g_- T^h_- = T^{gh}_- L^g_-$$

$\longrightarrow$ introduce orientation on edges



To each vertex v of the lattice we assign vertex operator defined by

$$A(v) = \frac{1}{|G|} \sum_{g \in G} L^g_{+,1} L^g_{+,2} L^g_{-,3} L^g_{-,4}$$

Similarly, define plaquette op.:

$$B(p) = \sum_{h_1 \cdots h_4 = 1} T^{h_1}_{-,1} T^{h_2}_{-,2} T^{h_3}_{+,3} T^{h_4}_{+,4}$$

All operators $A(v)$ and $B(p)$ commute with each other

$\rightarrow$ define Hamiltonian:

$$H = -\sum_v A(v) - \sum_p B(p)$$

Ground state is a stabilizer state satisfying

$(*)$ $\quad A(v)|\xi\rangle = |\xi\rangle, \; B(p)|\xi\rangle = |\xi\rangle \quad \forall \; v, p$

$\rightarrow$ excitations are identified by the violation of conditions $(*)$

The Hamiltonian is naturally gapped as $A(v)$ and $B(p)$ have discrete spectra

$\rightarrow$ quantum inf. is energetically protected

Example I: Abelian quantum double models

Let $G = \mathbb{Z}_d = \{0, 1, \dots, d-1\}$

→ we have: $g \cdot h = g + h \pmod d$

Next, consider lattice with square geometry and assign d-level spins on every edge

→ generalized Pauli operators

$$X = \sum_{h \in \mathbb{Z}_d} |h+1 \pmod d\rangle\langle h| ,$$

$$Z = \sum_{h \in \mathbb{Z}_d} \omega^h |h\rangle\langle h| , \quad \omega = e^{2\pi i/d}$$

For $d = 2$, we recover usual Pauli σ^x and σ^z

→ general (arbitrary d) commutation rels:

$$Z X = \omega X Z \qquad (**)$$

→ eigenstates of X-operator are:

$$|\tilde{g}\rangle = \frac{1}{\sqrt{d}} \sum_{h \in \mathbb{Z}_d} \omega^{gh} |h\rangle , \quad g = 0, \dots, d-1$$

with eigenvalues $\omega^{-g} = e^{-2\pi i g/d}$, $g \in \mathbb{Z}_d$

→ vertex and plaquette operators:

$$A(v) = X_1^\dagger X_2^\dagger X_3 X_4 , \quad B(p) = Z_1^\dagger Z_2 Z_3 Z_4^\dagger$$

both have eigenvalues ω^g, $g = 0, \dots, d-1$

Consider a general eigenstate $|\psi\rangle$ of all vertex and plaquette operators

→ vertex v, or plaquette p, is "unoccupied" if

$$A(v)|\psi\rangle = |\psi\rangle \quad \text{or} \quad B(p)|\psi\rangle = |\psi\rangle$$

→ • anyon e^g is associated with vertex v, if

$$A(v)|\psi\rangle = \omega^g|\psi\rangle,$$

• anyon m^h is associated with plaquette p, if

$$B(p)|\psi\rangle = \omega^h|\psi\rangle$$

• presence of both anyons is associated with composite particle $\varepsilon^{g,h}$

Specify Hamiltonian to be

$$H = -\left[\sum_v \sum_{h\in\mathbb{Z}_d} (A(v))^h + \sum_p \sum_{h\in\mathbb{Z}_d} (B(p))^h\right]$$

→ has anyonic vacuum $|\xi\rangle$ as ground state

→ assigns equal energy to all e^g quasiparticle excitations as $\sum_{h\in\mathbb{Z}_d} (A(v))^h$ act identically on each anyon e^g, $g\in\mathbb{Z}_d$.

analogously for excitations m^g

→ d^2 different particle species:

$$1,\ e^g,\ m^g,\ \varepsilon^{g,h} \ \forall\ g,h\in\mathbb{Z}_d$$

Fusion rules:

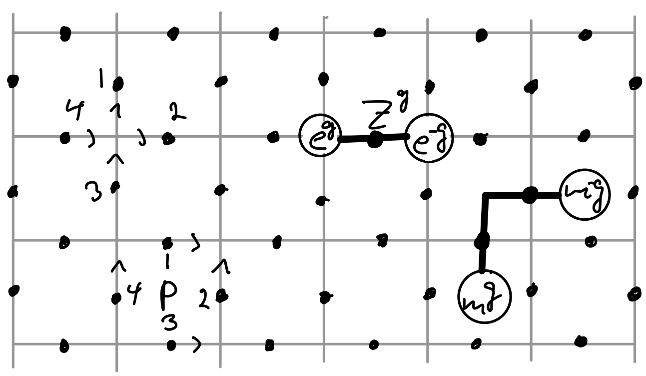
$$e^g \times e^h = e^{g+h \,(\mathrm{mod}\, d)}, \quad m^g \times m^h = m^{g+h\,(\mathrm{mod}\, d)},$$

$$e^g \times m^h = \varepsilon^{g,h}$$

Braiding: From the commutation relation (**) we deduce the R-matrix

$$\left(R^{\varepsilon^{g,h}}_{e^g m^h}\right)^2 = \omega^{gh}, \quad \omega = e^{2\pi i/d}$$

Generation of anyons is achieved by applying Z or X spin rotations to ground state $|\xi\rangle$:



$\rightarrow$ single spin rotations create particle-antip. pairs with positions determined by orientations of corresponding link

Example II: The non-Abelian $D(S_3)$ model

We take G to be simplest non-Abelian finite group: $G = S_3$

$$S_3 = \{e, c, c^2, t, tc, tc^2\}$$

identity   cyclic perm.   exchange of (1,2)

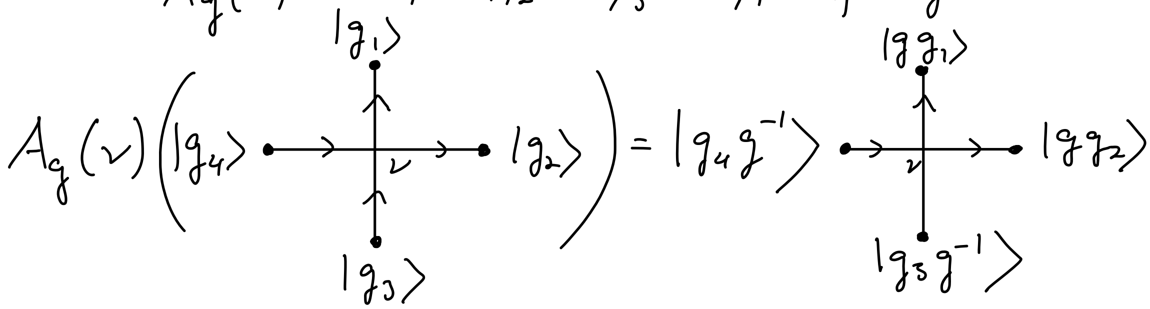
we have: $t^2 = c^3 = e, \quad tc = c^2 t$

$\longrightarrow |S_3| = 6$

Pick oriented two-dimensional square lattice $\longrightarrow$ assign 6-level spin spanned by states $|g\rangle$ to each edge

Define operators acting on vertex v by:

$$A_g(v) = L^g_{+,1} L^g_{+,2} L^g_{-,3} L^g_{-,4} \quad \text{for } g \in S_3$$



$\longrightarrow$ satisfy $[A_g(v), A_{g'}(v')] = 0 \;\forall\; v, v', g, g'$

$\longrightarrow$ build vertex operator $A(v)$ as

$$A(v) = \frac{1}{6}\left[A_e(v) + A_c(v) + A_{c^2}(v^2) + A_t(v) + A_{tc}(v) + A_{tc^2}(v)\right]$$

Similarly, define $B(p)$

$\longrightarrow$ stabilizer space consists of states with no quasi-particles:

$$A(v)|\xi\rangle = |\xi\rangle, \quad B(p)|\xi\rangle = |\xi\rangle$$

Hamiltonian is given by

$$H = -\sum_v A(v) - \sum_p B(p)$$

Define projectors onto quasiparticle occupations of vertices:

$$P_\Lambda(v) = \frac{1}{6}\left[A_e(v) + A_c(v) + A_{c^2}(v) - A_t(v) - A_{tc}(v) - A_{tc^2}(v)\right]$$

$$P_\Phi(v) = \frac{1}{3}\left[2A_e(v) - A_c(v) - A_{c^2}(v)\right]$$

$\longrightarrow$ have $P_X(v)P_{X'}(v) = 0$ for $X \neq X'$

$P_\Lambda(v)|\psi\rangle = |\psi\rangle \longrightarrow$ quasiparticle Λ at v

$P_\Phi(v)|\psi\rangle = |\psi\rangle \longrightarrow$ quasiparticle Φ at v

Creation operators:

$$W_\Lambda(s) = |e\rangle\langle e| + |c\rangle\langle c| + |c^2\rangle\langle c^2| - |t\rangle\langle t| - |tc\rangle\langle tc| - |tc^2\rangle\langle tc^2|$$

$$W_\Phi(s) = 2|e\rangle\langle e| - |c\rangle\langle c| - |c^2\rangle\langle c^2|$$

$\rightarrow$ can be checked by applying P_Λ and P_Φ

<u>Fusion rules</u>:

$$\Lambda \times \Lambda = 1, \quad \Lambda \times \Phi = \Phi, \quad \Phi \times \Phi = 1 + \Lambda + \Phi$$

$\longrightarrow$ Φ is non-Abelian anyon!

(there are more anyons in $D(S_3)$, but we focus here on closed sub-algebra $1, \Lambda, \Phi$)

Verification:

- $W_\Lambda(s) W_\Phi(s) = W_\Phi(s)$
- $W_\Phi(s) W_\Phi(s) = 4|e\rangle\langle e| + |c\rangle\langle c| + |c^2\rangle\langle c^2|$
  $= W_1(s) + W_\Lambda(s) + W_\Phi(s)$